RMSC 4003
Statistical Modeling in Financial Markets
Tutorial 6 Solution

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1 Summary of Chapter 1

- Typology of risks
- Tracking error
- Basis risk

2 Hedging with CAPM

This subsection summarizes what you learnt in RMSC2001 on forward and futures.

2.1 Forward Contracts and Futures Contracts

(1) The **forward price** of an asset today is the price at which you would agree to buy or sell the asset at a future time.

(2) The **value** of a forward contract is zero when you first enter into it. As time passes, the underlying asset price changes and the value of the contract may become positive or negative.

(3) When the interest rate is a **deterministic function of time**, the futures price **equals** to the forward price.

Pricing:

(a) $F_0 = S_0 e^{rT}$ if the asset provides **no income**.

(b) $F_0 = S_0 e^{(r-q)T}$ if the asset provides **known continuously compounded yield** q. (assuming all the income from the asset is reinvested in the asset)

2.2 Steps in hedging a portfolio with futures

The following are the steps in analyzing the performance of hedging a portfolio with futures. For an example, see lecture notes.

(1) Calculate the number of futures needed to short (if you are holding the portfolio) by $N = \beta \dfrac{P}{A}$.

(2) Calculate the theoretical futures price F_0 and F_1 by $F_0 = S_0 e^{(r-q)T_0}$ and $F_1 = S_1 e^{(r-q)T_1}$.

(3) Calculate the gain/loss from futures position by $(F_0 - F_1)NV$, where N is the number of futures and V is the contract multiplier (\$50 for HSI index futures and \$10 for mini-HSI futures.)

(4) Calculate the the portfolio return using CAPM: $\mu_P = r_f + \beta(\mu_M - r_f)$. For μ_M, you have to include the effect of dividend yield. For both interest rate and dividend yield, remember to multiply $\frac{m}{12}$ where m is the number of months in the period. For example (in lecture notes P.38),

$$\mu_P = r_f + \beta(\mu_M - r_f) = 0.04(\frac{3}{12}) + 1.5\left(\frac{900-1000}{1000} + 0.02(\frac{3}{12}) - 0.04(\frac{3}{12})\right) = -0.1475.$$

(5) Calculate the gain/loss from the portfolio.

(6) Calculate the net gain/loss.

3 Factor Model

3.1 One Factor Model

Conside the one factor model:

$$r_{it} = a_i + b_i F_t + \varepsilon_{it},$$

where r_{it} denotes the return of asset i at period t and ε_{it} are errors uncorrelated to the factor F_t. It is easy to see that

$$\begin{aligned} \mu_i &= a_i + b_i \mu_F \\ \sigma_i^2 &= bi^2 \sigma_F^2 + \sigma_{\varepsilon i}^2. \\ \sigma_{ij} &= b_i b_j \sigma_F^2 \\ b_i &= \frac{\text{Cov}(r_{it}, F_t)}{\sigma_F^2}. \end{aligned}$$

Now, suppose $F_t = r_{Mt}$, the market return and we have n assets. In this case, we have

$$b_i = \frac{\text{Cov}(r_{it}, r_{Mt})}{\sigma_M^2} = \beta_i.$$

This model also provides us a simple way to estimate the covariance matrix:

$$\begin{aligned} \Sigma &= \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix} \\ &= \begin{pmatrix} \beta_1^2 \sigma_M^2 + \sigma_{\varepsilon 1}^2 & \beta_1 \beta_2 \sigma_M^2 & \cdots & \beta_1 \beta_n \sigma_M^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_n \beta_1 \sigma_M^2 & \beta_n \beta_2 \sigma_M^2 & \cdots & \beta_n^2 \sigma_M^2 + \sigma_{\varepsilon n}^2 \end{pmatrix} \\ &= \sigma_M^2 \begin{pmatrix} \beta_1^2 & \beta_1 \beta_2 & \cdots & \beta_1 \beta_n \\ \vdots & \vdots & \ddots & \vdots \\ \beta_n \beta_1 & \beta_n \beta_2 & \cdots & \beta_n^2 \end{pmatrix} + \begin{pmatrix} \sigma_{\varepsilon 1}^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{\varepsilon n}^2 \end{pmatrix} \\ &= \boldsymbol{\beta} \boldsymbol{\beta}^{\boldsymbol{T}} \sigma_M^2 + \Sigma_\varepsilon, \end{aligned}$$

where

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \text{ and } \Sigma_\varepsilon = \begin{pmatrix} \sigma_{\varepsilon 1}^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_{\varepsilon n}^2 \end{pmatrix}.$$

So, the number of parameters needed to be estimated is reduced from $n(n+1)/2$ to $2n+1$.

Remark 3.1. If we write the factor model in vector notations:

$$\boldsymbol{r} = \boldsymbol{\alpha} + \boldsymbol{\beta} r_M + \boldsymbol{\varepsilon}.$$

Then

$$\begin{aligned} \Sigma &= \operatorname{Var}(\boldsymbol{r}) \\ &= \operatorname{Var}(\boldsymbol{\alpha} + \boldsymbol{\beta} r_M + \boldsymbol{\varepsilon}) \\ &= \operatorname{Var}(\boldsymbol{\beta} r_M) + \operatorname{Var}(\varepsilon) + 2\operatorname{Cov}(r_M \boldsymbol{\beta}, \varepsilon) \\ &= \boldsymbol{\beta}\boldsymbol{\beta}' \sigma_M^2 + \Sigma_\varepsilon + 0 \\ &= \boldsymbol{\beta}\boldsymbol{\beta}' \sigma_M^2 + \Sigma_\varepsilon. \end{aligned}$$

3.2 Estimating Factor Models

There are two main approahces in estimating a factor model: time series approach and cross-sectional approach.

3.2.1 Time Series Approach

$$r_{it} = a_i + b_i F_t + \varepsilon_{it}, \quad t = 1, \dots, T.$$

For example, you collect the data r_{it}, F_t for $t = 1, \dots, T$, where F_t is a factor like GDP, that is a common to all stock. Then, you will use a time series regression to estimate the factor sensitivities a_i and b_i.

3.2.2 Cross-Sectional Approach

$$r_{it} = a_i + b_{it} F_t + \varepsilon_{it}, \quad i = 1, \dots, n.$$

For example, you collect the data r_{it}, b_{it} for $= 1, \dots, n$, where b_{it} is the sensitivity or attribute like dividend yield, that is asset dependent. Then, you will use a cross-sectional regression to estimate the factor value F_t.

4 Exercises

Example 4.1 (Efficient Frontier; Tangency Portfolio)**.** Suppose the mean and the standard deviation of the tangency portfolio are 6% and 30% respectively, and the risk-free rate $r_f = 3\%$.

(a) Is there a portfolio that has $\mu = 4\%$ and $\sigma = 10\%$?

(b) Give an impossible combination of expected return and standard deviation in this setting.

Solution. (a) Note that the slope of the efficient frontier is $(6-3)/30 = 0.1$. Since $(4-3)/10 = 0.1$, we can form a portfolio with the required expected return and standard deviation.

(b) $\mu = 5\%$ and $\sigma = 10\%$ implies $(5-3)/10 = 0.2 > 0.1$. Impossible.

Example 4.2 (Factor Model)**.** Suppose there are three assets with return r_1, r_2, r_3 and a market portfolio r_M. If the weights in the market portfolio is (w_1, w_2, w_3).

(a) Write down the covariance of r_1 and r_M in terms of $w_1, w_2, w_3, \sigma_{11}, \sigma_{12}, \sigma_{13}$.

(b) Suppose the three assets follow the single factor model:

$$r_i = r_f + \beta_i(r_M - r_f) + e_i$$

for $i = 1, 2, 3$.

(i) Suppose $\beta_1 = 1, \beta_2 = 0.8$ and $\beta_3 = 0.5$. If short-selling is not allowed, can you form a portfolio which contains idiosyncratic risk only?

(ii) Suppose $\beta_1 = 1, \beta_2 = 0.8$ and $\beta_3 = 0.5$. If short-selling is allowed, can you form a portfolio which contains idiosyncratic risk only?

(iii) Suppose $\beta_1 = 1, \beta_2 = 0.8$ and $\beta_3 = -0.5$. If short-selling is not allowed, can you form a portfolio which contains idiosyncratic risk only?

Solution. (a) Since $r_M = w_1r_1 + w_2r_2 + w_3r_3$, $\mathrm{Cov}(r_1, r_M) = w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13}$

(b) (i) $v_1 + v_2 + v_3 = 1$ and $v_1 + 0.8v_2 + 0.5v_3 = 0$ implies $0.2v_2 + 0.5v_3 = 1$. If short-selling is not allowed, we cannot form a portfolio which contains idiosyncratic risk only.

(ii) Yes. For example, set $v_2 = 0$. Then $v_3 = 2$ and so $v_1 = -1$.

(iii) Yes. Now we have $v_1 + v_2 + v_3 = 1$ and $v_1 + 0.8v_2 - 0.5v_3 = 0$. Thus, $0.2v_2 + 1.5v_3 = 1$. Therefore, we can take $v_2 = 0$ and $v_3 = 2/3$. Then $v_1 = 1/3$.

The CAPM is a pricing model. However, the standard CAPM formula does not contain prices explicitly. To see why the CAPM is called a pricing model, we may look at the following example.

Example 4.3 (CAPM)**.** Suppose that an asset is purchased at price P and later sold at price Q. Then the rate of return is $R = \dfrac{Q-P}{P}$. Denote $\overline{Q}$ to be the expected value of Q.

(a) Using the security market line of CAPM, show that

$$P = \frac{\overline{Q}}{1 + r_f + \beta(\mu_M - r_f)},$$

where μ_M denotes the mean return rate of the market, r_f, denotes the risk-free rate, and β denotes the beta of the underlying asset used in CAPM.

(b) Suppose σ_{QM} is known. Show that the above equation can be written as

$$P = \frac{\overline{Q} - \lambda\sigma_{QM}}{1 + r_f},$$

where $\lambda = (\mu_M - r_f)/\sigma_M^2$.

(c) Suppose the Calvin is considering investing in bond, which has a face value of \$100, with an annual coupon rate of 8%. Suppose that this bond has a 70% chance of paying the investor plus coupon in full in one period, and a 30% not paying the full face value nor the coupon, but only \$50 as underwriting fee to the investor. Assume that $\mathrm{Cov}(Q, r_M) = 7$, where r_M is the return rate of the market, $r_f = 0.1$, $\mu_M = 0.2$, $\sigma_M^2 = 0.09$. Determine the price P of this bond at the initial period.

(d) Evaluate R.

Solution. (a)

$$\begin{aligned}
\overline{R} - r_f &= \beta(\mu_M - r_f) \\
\frac{\overline{Q} - P}{P} &= r_f + \beta(\mu_M - r_f) \\
\frac{\overline{Q}}{P} &= 1 + r_f + \beta(\mu_M - r_f)
\end{aligned}$$

So, we have

$$P = \frac{\overline{Q}}{1 + r_f + \beta(\mu_M - r_f)}.$$

(b) First, observe that $\sigma_{RM} = \mathrm{Cov}(R, r_M) = \mathrm{Cov}(\frac{Q-P}{P}, r_M) = \frac{\sigma_{QM}}{P}$. From part (a), we have

$$\begin{aligned}
(1 + r_f)P + \beta(\mu_M - r_f)P &= \overline{Q} \\
(1 + r_f)P + \frac{\sigma_{RM}}{\sigma_M^2} P\lambda\sigma_M^2 &= \overline{Q} \\
(1 + r_f)P + \sigma_{RM} P\lambda &= \overline{Q} \\
(1 + r_f)P + \lambda\sigma_{QM} &= \overline{Q}.
\end{aligned}$$

So, we have

$$P = \frac{\overline{Q} - \lambda\sigma_{QM}}{1 + r_f}.$$

(c) $\overline{Q} = (108)(0.7) + (50)(0.3) = 90.6$. $P = \dfrac{90.6 - \frac{0.2-0.1}{0.09} \times 7}{1 + 0.1} = 75.29$.

(d) $R = \dfrac{108 - 75.29}{75.29} = 0.4344$ with probability 0.7 and $R = \dfrac{50 - 75.29}{75.29} = -0.3359$ with probability 0.3.